BGSM/CRM
AL&DNN

Geometrical Physics
Symmetries and Noether's theorem

S. Xambó

UPC & IMTech

2/11/2021

Abstract. Lagrangian and Hamiltonian mechanics. Symmetries in physical systems and conserved quantities. Noether's theorem.

References:

[1] (bronstein-bruna-cohen-velickovic-2021)
[2] (cohen-2021)
[3] (lavor-xambo-zaplana-2018)
[4] (frankel-2011)
[5] (folland-2008)

For Noether's theorem:

[6] (kosmann-2011)
[7] (neuenschwander-2011)
See also [8, page 786].

Geometrical physics

Lagrangian analytical approach
Hamiltonian formalism
Symmetries in the physical systems
Noether's theorem

Lagrangian analytical approach

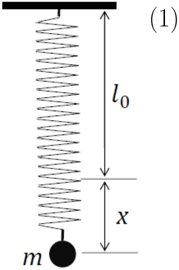
- Joseph Louis Lagrange (1736-1813): *Mécanique analytique* (1788).

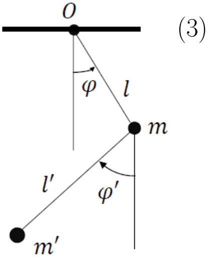
- Wrote the evolution equations of a mechanical system in terms of arbitrary *generalized coordinates* q_j (parameters specifying the configuration of the system):

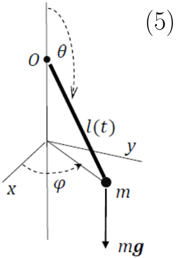
$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} = \frac{\partial (T-V)}{\partial q_j} \; (j = 1, \ldots, n).$$

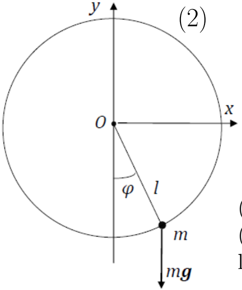
- Has had a major influence in the development of *differential geometry* (manifolds).

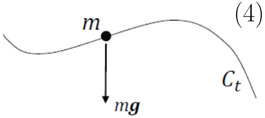
- The Lagrangian method has played a key role not only in Mechanics, but also in the *field theory* (both *classical* and *quantum* fields).

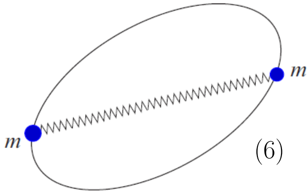












(1) Harmonic oscillator. (2) Simple pendulum. (3) Duble pendulum. (4) Mas sliding on a moving curve. (5) Spherical pendulum of variable lenght. (6) Two masses moving on a curve and connected with a spring

- $m_1, \ldots, m_N \in \mathbf{R}_{++}$: *point masses*
- $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N$: *positions of the point masses*
- $\boldsymbol{v}_j = \frac{d\boldsymbol{r}_j}{dt} = \dot{\boldsymbol{r}}_j$: *velocity of* m_j
- $\boldsymbol{p}_j = m_j \boldsymbol{v}_j$: (linear) *moment of* m_j
- $\boldsymbol{F}_j$: *force acting on* m_j: $\boldsymbol{F}_j = m_j \boldsymbol{a}_j = m_j \dot{\boldsymbol{v}}_j = \dot{p}_j$
- $f_\alpha(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n, t) = 0$, $\alpha = 1, \ldots, m$: *constraints*
- $\mathcal{X}_t$: *configuration space at time* t:

$$\mathcal{X}_t = \{(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) \in E_3^N : f_\alpha(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n, t) = 0, \alpha \in [m]\}$$

Note: Simply $\mathcal{X}$ if the constraints do not depend on t.

Note: Depending on the scale, a point masses can be a *galaxies*, *stars in a galaxy*, *planets around a star* (like the *solar system*), *molecules* (in *solid bodies*, deformable or rigid, in *liquids*, or in *gases*). And they can be simple idealized examples as in the illustrations.

$\boldsymbol{F}_k = \sum_{j\neq k} G \frac{m_j m_k}{|\boldsymbol{r}_j - \boldsymbol{r}_k|^3}(\boldsymbol{r}_j - \boldsymbol{r}_k)$,

G=6.67×10^{-11} N m^2 Kg^{-2}.

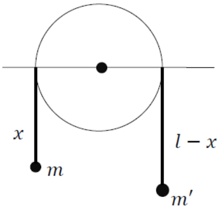
There are no constraints.

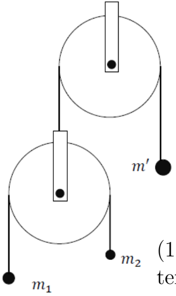
The constraints are said to be *holonomic* if the positions $\boldsymbol{r}_j$ can be expressed (locally in $\mathcal{X}_t$) as functions $\boldsymbol{r}_j = \boldsymbol{r}_j(q,t)$, where $q = (q_1, \ldots, q_n) \in U$, $U \subseteq \mathbf{R}^n$ open, such that

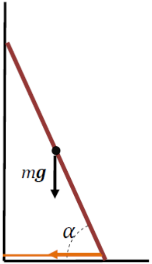
$$(q,t) \mapsto (\boldsymbol{r}_1(q,t), \ldots, \boldsymbol{r}_n(q,t), t)$$

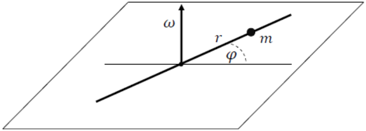
is a diffeormorfism of U with an open set $U' \subseteq \mathcal{X}_t$.

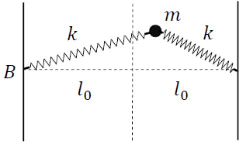
In other words, $\mathcal{X}_t$ is a manifold of dimension n.

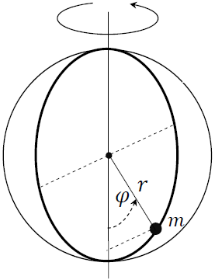












(1) and (2) Simple and double Atwood machines. (3) Stattics of a ladder: tension of the rope connecting its foot to the wall. (4) Mass aliding on a straight rod that is rotating about a perpendicular line. (5) Mass connected to two fixed points by springs of the same elastic constant. (6) Mass sliding on a circumference that is turning about a vertical diameter.

$\mathcal{S} \subseteq E_3^N \times E_3^N \times \mathbf{R}$.

Its points $(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_N, t)$ are such that

$(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N, t) \in \mathcal{X}_t$ and $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N)$ are the possible velocities allowed by the contraints.

- $\sum_j \boldsymbol{\partial}_j f \cdot \boldsymbol{v}_j + \partial_t f_\alpha = 0$ $(\boldsymbol{\partial}_j = \frac{\partial}{\partial \boldsymbol{r}_j},\ \partial_t = \frac{\partial}{\partial t})$.
- $\boldsymbol{v}_j = \dot{\boldsymbol{r}}_j = \sum_k (\partial_k \boldsymbol{r}_j)\dot{q}_k + \partial_t \boldsymbol{r}_j$ $(\partial_k = \frac{\partial}{\partial q_k})$ [∗]
- $(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = (q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_N, t)$: *local coordinates of* $\mathcal{S}$.

Lemma. (1) $\dot{\partial}_k \dot{\boldsymbol{r}}_j = \partial_k \boldsymbol{r}_j$ $(\dot{\partial}_k = \frac{\partial}{\partial \dot{q}_k})$. (2) $\frac{d}{dt}\partial_k \boldsymbol{r}_j = \partial_k \dot{\boldsymbol{r}}_j$.

(1) is a direct consequence of [∗]. (2) follows from the chain rule and Schwarz's theorem on second derivatives.

$$T = \sum_{j=1}^{N} \frac{1}{2} m_j \boldsymbol{v}_j^2 = \sum_{j=1}^{N} \frac{1}{2} m_j \left(\sum_k (\partial_k \boldsymbol{r}_j) \dot{q}_k + \partial_t \boldsymbol{r}_j \right)^2$$
$$= T_0 + T_1 + T_2,$$

$$T_0 = \sum_{j=1}^{N} \frac{1}{2} m_j (\partial_t \boldsymbol{r}_j)^2$$

$$T_1 = \sum_{j=1}^{N} m_j \left(\sum_k (\partial_k \boldsymbol{r}_j) \dot{q}_k \right) \cdot \partial_t \boldsymbol{r}_j$$

$$T_2 = \sum_{j=1}^{N} \frac{1}{2} m_j \left(\sum_k (\partial_k \boldsymbol{r}_j) \dot{q}_k \right)^2$$

Note. $T = T_2$ if the constraints are not dependent on t (*scleronomous constraints*)

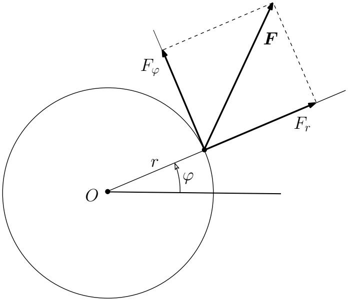
- $Q_k = \sum_{j=1}^{N} \boldsymbol{F}_j \cdot \partial_k \boldsymbol{r}_j$ $(k = 1, \ldots, n)$, $Q_t = \sum_{j=1}^{N} \boldsymbol{F}_j \cdot \partial_t \boldsymbol{r}_j$ (*generalized forces*).

Example (Generalized forces on a point mass m moving in $\mathbf{R}^2$ with respect to polar coordinates r, φ). We have $x = r\cos\varphi$, $y = r\sin\varphi$, hence $\boldsymbol{r} = r(\cos\varphi, \sin\varphi)$. and

$$Q_r = \boldsymbol{F} \cdot \partial_r \boldsymbol{r} = \boldsymbol{F} \cdot (\cos\varphi, \sin\varphi) = \boldsymbol{F} \cdot \widehat{\boldsymbol{r}} = F_r,$$

$$Q_\varphi = \boldsymbol{F} \cdot \partial_\varphi \boldsymbol{r} = \boldsymbol{F} \cdot (-r\sin\varphi, r\cos\varphi) = r\boldsymbol{F} \cdot \widehat{\boldsymbol{\varphi}} = rF_\varphi,$$

where $\widehat{\boldsymbol{r}} = \boldsymbol{r}/r$ and $\widehat{\boldsymbol{\varphi}} = \widehat{\boldsymbol{r}}^{\perp}$, and hence F_r and F_φ are the components of $\boldsymbol{F}$ with respect to the orthonormal basis $\widehat{\boldsymbol{r}}, \widehat{\boldsymbol{\varphi}}$.



Theorem. The evolution of a holonomic mechanical system is governed by the equations

$$d_t\dot{\partial}_k T - \partial_k T = Q_k \; (k = 1, \ldots, n, \; d_t = \tfrac{d}{dt}).$$

Proof. If in the infinitesimal time interval dt the position vectors change by $d\boldsymbol{r}_j$, the work done by the forces is

$$\begin{aligned} W &= \textstyle\sum_j \boldsymbol{F}_j \cdot d\boldsymbol{r}_j = \sum_j \boldsymbol{F}_j \cdot (\sum_k (\partial_k \boldsymbol{r}_j) dq_k + (\partial_t \boldsymbol{r}_j) dt) \\ &= \textstyle\sum_k (\sum_j \boldsymbol{F}_j \cdot \partial_k \boldsymbol{r}_j) dq_k + (\sum_j \boldsymbol{F}_j \cdot \partial_t \boldsymbol{r}_j) dt \\ &= \textstyle\sum_k Q_k dq_k + Q_t dt. \end{aligned}$$

On the other hand we have $\boldsymbol{F}_j = m_j \ddot{\boldsymbol{r}}_j$, and we can write:

$$
\begin{aligned}
W &= \sum_j m_j \ddot{\boldsymbol{r}}_j \cdot d\boldsymbol{r}_j \\
&= \sum_j m_j \ddot{\boldsymbol{r}}_j \cdot (\sum_k (\partial_k \boldsymbol{r}_j) dq_k + (\partial_t \boldsymbol{r}_j) dt) \\
&= \sum_{j,k} m_j \big(d_t(\dot{r}_k \cdot \partial_k \boldsymbol{r}_j) - \dot{r}_j \cdot d_t \partial_k \boldsymbol{r}_j\big) dq_k + Q_t dt \\
&= \sum_{j,k} m_j \big(d_t(\dot{r}_k \cdot \dot{\partial}_k \dot{r}_j) - \dot{r}_j \cdot d_t \partial_k \boldsymbol{r}_j\big) dq_k + Q_t dt \\
&= \sum_{j,k} \big(d_t \dot{\partial}_k (\tfrac{1}{2} m_j \dot{r}_j^2) - \partial_k (\tfrac{1}{2} m_j \dot{r}_j^2)\big) dq_k + Q_t dt \\
&= \sum_k \big(d_t \dot{\partial}_k T - \partial_k T\big) dq_k + Q_t dt
\end{aligned}
$$

Now the claim follows on equating the coefficients of dq_k in both expressions. □

Remark. If there are no constraints and we use the cartesian coordinates of the $\boldsymbol{r}_j$, the Lagrange equations are equivalent to Newton's equations.

Evolution of a particle in a plane using polar coordinates

In cartesian coordinates x, y, the kinetic enerby is $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$. In polar coordinates r, φ, we have $x = r\cos\varphi$, $y = r\sin\varphi$, and a straightforward computation shows that $T = \frac{1}{2}m(\dot{r}^2 + (r\dot{\varphi})^2)$.

We also know that the generalized forces with respect to polar coordinates are $Q_r = F_r$ and $Q_\varphi = F_\phi$ (the components of $\boldsymbol{F}$ with respect to the orthonormal basis $\widehat{\boldsymbol{r}}, \widehat{\boldsymbol{\varphi}}$).

$\dot{\partial}_r T$	$d_t\dot{\partial}_r T$	$\partial_r T$	Eq_r
$m\dot{r}$	$m\ddot{r}$	$mr\dot{\varphi}^2$	$m\ddot{r} - mr\dot{\varphi}^2 = F_r$

$\dot{\partial}_\varphi T$	$d_t\dot{\partial}_\varphi T$	$\partial_\phi T$	Eq_φ
$mr^2\dot{\varphi}$	$2mr\dot{r}\dot{\varphi} + mr^2\ddot{\varphi}$	0	$mr^2\ddot{\varphi} + 2mr\dot{r}\dot{\varphi} = rF_\varphi$

The constraints $f_\alpha(\boldsymbol{r}_1,\dots,\boldsymbol{r}_N,t)$ $(\alpha=1,\dots,m)$ are said to be *ideal* if for any state there exist $\lambda_\alpha\in\mathbf{R}$ such that

$$\boldsymbol{R}_j=\sum_\alpha\lambda_\alpha\boldsymbol{\partial}_j f_\alpha,$$

where $\boldsymbol{R}_j$ is the resultant of the *constraining forces* on m_j. The λ_α may depend on $(\boldsymbol{r}_1,\dots,\boldsymbol{r}_N,t)$, but they should not depend on j.

Remark. The usefulness of the concept of ideal constraints comes, on the one hand, from the fact that it *holds in many circumstances* (at least in the first approximation) and, on the other, that the contribution of the *constraining forces in the calculation of generalized forces is* 0 for ideal constraints.

Example. The constraint of a simple pendulum is $f(\boldsymbol{r})-l^2=0$. The constrining force is proportonal to $\boldsymbol{r}$, say $\boldsymbol{R}=\mu\boldsymbol{r}$. On the other hand $\partial_{\boldsymbol{r}}f=2\boldsymbol{r}$, and hence $\boldsymbol{R}=\frac{1}{2}\mu\partial_{\boldsymbol{r}}f$.

Double pendulum. If and $\boldsymbol{r}$ and $\boldsymbol{r}'$ are the position vectors of the two masses m and m' with respect to supension point O of the first pendulum, the constraining forces $\boldsymbol{R}$ (on m) and $\boldsymbol{R}'$ (on m') have the form (using Newton's third law)

$$\boldsymbol{R} = \mu\boldsymbol{r} + \mu'(\boldsymbol{r} - \boldsymbol{r}'),\ \boldsymbol{R}' = \mu'(\boldsymbol{r}' - \boldsymbol{r}),\ \mu, \mu' \in \mathbf{R}$$

The constraints are

$$f = \boldsymbol{r}^2 - l^2,\ f' = (\boldsymbol{r}' - \boldsymbol{r})^2 - l'^2 = 0$$

and the conclusion is clear from the following table:

∂	$\partial_{\boldsymbol{r}}$	$\partial_{\boldsymbol{r}'}$
f	$2\boldsymbol{r}$	0
f'	$2(\boldsymbol{r} - \boldsymbol{r}')$	$2(\boldsymbol{r}' - \boldsymbol{r})$
$\boldsymbol{R}$	$\mu\boldsymbol{r} + \mu'(\boldsymbol{r} - \boldsymbol{r}')$	$\mu'(\boldsymbol{r}' - \boldsymbol{r})$

Particle moving with no friction on the variable surface. Let $f(\boldsymbol{r},t)=0$ be the moving surface. If the particle moves with no friction, the constraining force $\boldsymbol{R}$ must be orthogonal to $\mathcal{X}_t=\{\boldsymbol{r}\in E_3 : f(\boldsymbol{r},t)=0\}$ and hence $\boldsymbol{R}=\lambda\partial_{\boldsymbol{r}}f$, which means that the constraint is ideal.

Rigid bodies. A *rigid body* can be thought as a set of point masses $m_1,\ldots,m_N$ with constraints

$$f_{ij}=(\boldsymbol{r}_i-\boldsymbol{r}_j)^2-d_{ij}^2=0, \text{ where } d_{ij} \text{ are constants.}$$

The constraining force that m_i exerts on m_j has the form $\boldsymbol{R}_{ij}=\mu_{ij}(\boldsymbol{r}_i-\boldsymbol{r}_j)$, and $\mu_{ij}=\mu_{ji}$ by Newton's third law. Let $\lambda_{ij}=-\mu_{ij}/4$. Then we have

$$\begin{aligned}\textstyle\sum_{ij}\lambda_{ij}(\boldsymbol{\partial}_k f_{ij}) &= 2\textstyle\sum_j\lambda_{kj}(\boldsymbol{r}_k-\boldsymbol{r}_j)+2\textstyle\sum_i\lambda_{ik}(\boldsymbol{r}_k-\boldsymbol{r}_i)\\ &= \textstyle\sum_i\mu_{ik}(\boldsymbol{r}_i-\boldsymbol{r}_k)=\textstyle\sum_i\boldsymbol{R}_{ik}=\boldsymbol{R}_k.\end{aligned}$$

Theorem. In a holonomic system, the cotribution of the constraining forces to the generalized forces is 0.

Proof. If the constraints are ideal, then $\boldsymbol{R}_i = \sum_\alpha \lambda_\alpha \boldsymbol{\partial}_i f_\alpha$ ($\lambda_\alpha \in \mathbf{R}$), and their contribution of to the generalized force Q_k is

$$\sum_i \boldsymbol{R}_i \cdot \partial_k \boldsymbol{r}_i = \sum_{i,\alpha} \lambda_\alpha \boldsymbol{\partial}_i f_\alpha \cdot \partial_k \boldsymbol{r}_i = \sum_\alpha \lambda_\alpha \partial_k f_\alpha = 0,$$

because f_α is, for a fixed t, identically 0 as a function of the $q_1, \ldots, q_n$. □

Corollary. The Lagrange equations of a holonomic system with ideal constraints have the form

$$d_t \dot{\partial}_k T - \partial_k T = Q'_k, \text{ where } Q'_k = \textstyle\sum_i (\boldsymbol{F}_i - \boldsymbol{R}_i) \cdot \partial_k \boldsymbol{r}_i. \quad \square$$

The forces $\boldsymbol{F}'_i = \boldsymbol{F}_i - \boldsymbol{R}_i$ are the *net forces* acting on the system. They are the sum of the *interaction forces* between the particles (like the gravitational forces) and the *applied* or *external* forces (like gravity if the particles are placed in a gravitational field).

Henceforth, by *mechanical system* we will understand a *holonomic mechanichal system*, the forces will $\boldsymbol{F}_j$ will be the net forces, and $Q_k = \sum_j \boldsymbol{F}_j \cdot \partial_k \boldsymbol{r}_i$ the generalized forces. By the corollary above, these systems are governed by the equations

$$d_t\dot{\partial}_k T - \partial_k T = Q_k.$$

Remark (The d'Alambert principle). If the constraints are time-dependent, the constraining forces *can do work*. In fact, if $\boldsymbol{R}_i = \sum_\alpha \lambda_\alpha \boldsymbol{\partial}_i f_\alpha$, then the *power* produced by the $\boldsymbol{R}_i$ is, as a consequence of the chain rule,

$$\sum_i \boldsymbol{R}_i \cdot \dot{r}_i = -\sum_\alpha \lambda_\alpha \partial_t f_\alpha.$$

In particular, if the constraints do not depend on t, then the constraining forces do no work. This is known as the *d'Alembert principle* (of virtual work).

The forces $\boldsymbol{F}_j$ are said to be *conservative* if there exists a function $V = V(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N, t)$ (called the *potential*) such that

$$\boldsymbol{F}_i = -\boldsymbol{\partial}_i V.$$

In this case the mechanical system is said to be *conservative*.

Example. The function $V = G\sum_{i\neq j} m_i m_j / |\boldsymbol{r}_i - \boldsymbol{r}_j|$ is a potential for the newtonian gravitational forces

$$\boldsymbol{F}_i = G\sum_{j\neq i}(\boldsymbol{r}_i - \boldsymbol{r}_j)/|\boldsymbol{r}_i - \boldsymbol{r}_j|^3.$$

Indeed, from $\boldsymbol{\partial}(1/r) = -r^{-3}\boldsymbol{r}$,

$$\boldsymbol{\partial}_i(1/|\boldsymbol{r}_i - \boldsymbol{r}_j|) = -(\boldsymbol{r}_i - \boldsymbol{r}_j)/|\boldsymbol{r}_i - \boldsymbol{r}_j|^3,$$

and this implies the claim.

Lemma. If we express V as a functions of the generalized coordinates $q_1, \ldots, q_N$, then $Q_k = \sum_i -\boldsymbol{\partial}_i V \cdot \partial_k \boldsymbol{r}_i = -\partial_k V$.

For a conservative system, the function $L = T - V : \mathcal{S} \to \mathbf{R}$ is called the *lagrangian* of the system.

Theorem (Euler-Lagrange). A conservative mechanical system is governed by the equations (*Euler-Lagrange equations*)

$$d_t \dot{\partial}_k L - \partial_k L = 0 \ (k = 1, \ldots, n). \qquad [*]$$

Proof. Since V does not depend on the $\dot{q}_j$, $d_t \dot{\partial}_k L = d_t \dot{\partial}_k T$. On the other hand, -$\partial_k L = -\partial_k T + \partial_k V = -\partial_k T - Q_k$, and hence the equations [*] are equivalent to $d_t \dot{\partial}_k T - \partial_k T = Q_k$. □

Definition. A holonomic system is said to be *lagrangian* if there exists a function $L = L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ such that its evolution is governed by the equations

$$d_t \dot{\partial}_k L - \partial_k L = 0 \ (k = 1, \ldots, n).$$

Clearly, a conservative mechanical system is lagrangian, with lagrangian function $L = T - V$.

Observables of a Lagrangian system. Energy

Observables, conserved quantities and conjugate momenta

Example: Kepler's second law

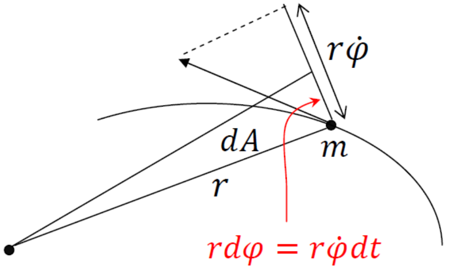
Conditions for the conservation of energy

- *Observable*: A function $f : \mathcal{S} \to \mathbf{R}$, $f = f(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$. If f does not depend on $\dot{\boldsymbol{q}}$, we say that f is a *configuration observable*.
- A *conserved quantity*, or *first integral*, is an observable f such that $\dot{f} = 0$. This means that f remains constant during the temporal evolution of the system.
- *Conjugate momenta*: In a lagrangian system with lagrangian L, they are the observables $p_k = \dot{\partial}_k L$. They are also called *canonical momenta*.
- In rectangular cartesian coordinates, $\dot{\boldsymbol{\partial}}_k L = \partial_{\dot{\boldsymbol{r}}_k} L = m_k \dot{\boldsymbol{r}}_k = \boldsymbol{p}_k$.
- A generalized coordinate q_k is *cyclic* if L does not depend on q_k.
- *Example*. In polar coordinates, the lagrangian of a point mass m moving in $\mathbf{R}^2$ under a central potential $V(r)$ is $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}) - V(r)$. Thus φ is cyclic.

- If q_k is a cyclic cordinate of a lagrangian system, them p_k is a conserved quantity.

$$\dot{p}_k = d_t \dot{\partial}_k L = \partial_k L = 0. \qquad \square$$

Example. With the same assumptions and notations as in the example in the previous page, φ is a cyclic coordinate of $L = m(\dot{r}^2 + r^2\dot{\varphi}) - V(r)$ and its conjugate momentum is $p_\varphi = \partial_{\dot{\varphi}} L = mr^2\dot{\varphi}$.



So this is a conserved quantity. Since $r\dot{\varphi}$ is the tranversal velocity, $mr^2\dot{\varphi} = r(mr\dot{\varphi})$ is the angular momentum h of m with respect to the origin. So h is a conserved quantity.

If A is the area swept by r, we hav

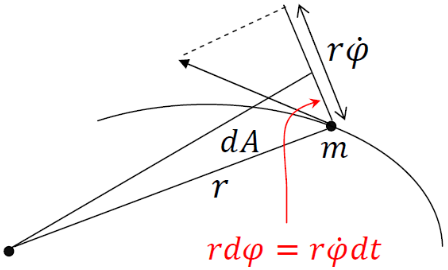
$$2dA = r(rd\varphi) = r^2 d\varphi.$$

Consequently,

$$\dot{A} = \frac{1}{2} r^2 \dot{\varphi} = h/2m$$

is constant.

This is *Kepler's second law* for a mass m in a central potential: *the areolar velocity* (namely $\dot{A}$) *is constant*.



- Consider a holonomic system and let V be *a potential for the conservative forces*.
- $\boldsymbol{F}'_j$: the non-conservative force on m_j, hence $\boldsymbol{F}_j = \boldsymbol{F}'_j - \boldsymbol{\partial}_j V$.
- $Q'_1, \ldots, Q'_n$: *generalized forces produced by the* $\boldsymbol{F}'_j$.
- $W' = \sum_k Q'_q \dot{q}_k$: *generalized power of the non-coservative forces*.
- We have seen that $T = T_2 + T_1 + T_0$, where T_j is *homogeneous* of degre j in the $\dot{q}_k$.
- $\sum_k \dot{q}_k(\dot{\partial}_k T) = 2T_2 + T_1$. (Use *Euler's lemma*: if $f = f(x_1, \ldots, x_n)$ is homogeneous of degree m, then $\sum_k x_k \partial_k f = mf$).

- The observable $E = T + V$ is the *mechanical energy* of the system, and $L = T - V$ the *lagrangian*

Theorem. $\dot{E} = W' - \partial_t L + d_t(T_1 + 2T_0)$.

$$\begin{aligned}\dot{T} &= \textstyle\sum_k (\partial_k T)\dot{q}_k + (\dot{\partial}_k T)\ddot{q}_k + \partial_t T \\ &= \textstyle\sum_k d_t((\dot{\partial}_k T)\dot{q}_k) + \sum_k (\partial_k T - d_t \dot{\partial}_k T)\dot{q}_k + \partial_t T \\ &= d_t(2T_2 + T_1) + \textstyle\sum_k (\partial_k V - Q'_k)\dot{q}_k + \partial_t T \\ &= 2\dot{T} - d_t(T_1 + 2T_0) + d_t V - \partial_t V - W' + \partial_t T \\ &= \dot{T} + \dot{E} + \partial_t L - W' - d_t(T_1 + 2T_0),\end{aligned}$$

and from this the claim follows immediately. □

Corollary. (1) If the constraints do not depend on t, $\dot{E} = \partial_t V + W'$. (2) If in addtion V does not depend on t, then $\dot{E} = W'$. (3) Finally, *the mechanical energy is conserved for holonomic conservative systems whose constraints and potential do not depend on* t. □

Remark. The non-conservative forces for which $W' < 0$ are called *dissipative forces*.

If $W' = 0$, they are called *gyroscopic*.

The *Coriolis forces*, due to the rotation of the Earth, are gyroscopic: they do no work because they are perpendicular to the velocity of particles.

Hamilton's formalism

The Hamiltonian
Legendre transformation
Hamilton's equations

The *Hamiltonian* of a Lagrange system is the observable

$$H = \sum_k p_q \dot{q}_k - L.$$

Lemma. $H = T_2 - T_0 + V = E - (T_1 + 2T_0)$.

Proof. First note that

$$\sum_k p_k \dot{q}_k = \sum_k (\dot{\partial}_k L)\dot{q}_k = 2T_2 + T_1 \text{ (by Euler's lemma).}$$

Therefore,

$$H = 2T_2 + T_1 - (T_2 + T_1 + T_0 - V) = T_2 - T_0 + V,$$

which is the first expression. Now $T_2 - T_0 = T - (T_1 + 2T_0)$, hence

$$T_2 - T_0 + V = T + V - (T_1 + 2T_0) = E - (T_1 + 2T_0),$$

which is the second expression. □

Corollary. If $T = T_2$, which happens if the constraints do not depend on t, then $H = E$. □

The *Legendre transformation* is the map

$$(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \mapsto (\boldsymbol{q}, \boldsymbol{p}, t),\ \boldsymbol{p} = \partial_{\dot{\boldsymbol{q}}} L.$$

Example. The Lagrangian of a harmonic multioscillator is

$$L = \sum_j \tfrac{1}{2} m_j \dot{q}_j^2 - \sum_j \tfrac{1}{2} \kappa_j q_j^2.$$

In this case $\partial_{\dot{\boldsymbol{q}}} L = (m_1 \dot{q}_1, \ldots, m_n \dot{q}_n)$ and hence the Legendre transformation is

$$(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) \mapsto q_1, \ldots, q_n, m_1 \dot{q}_1, \ldots, m_n \dot{q}_n, t).$$

If the Legendre transformation is a diffeomorphism (as for example in the harmonic multioscillator), we say that the mechanical system is *hamiltonian*.

Theorem. The evolution of a Hamiltonian system is governed by *Hamilton's equations*:

$$\dot{\boldsymbol{q}} = \partial_{\boldsymbol{p}} H,\ \ \dot{\boldsymbol{p}} = -\partial_{\boldsymbol{q}} H.$$

Moreover, the following relations hold: $d_t H = \partial_t H = -\partial_t L$.

Proof. $dH = \sum_k (\partial_k H) dq_k + \sum_k (\partial'_k H) dp_k + (\partial_t H) dt$ $(\partial'_k = \partial_{p_k})$.

Using the definition $H = \sum_k p_q \dot{q}_k - L$, we get

$$\begin{aligned} dH &= \textstyle\sum_k \dot{q}_k dp_k + \sum_k p_k d\dot{q}_k - \sum_k (\partial_k L) dq_k - \sum_k (\dot{\partial}_k L) d\dot{q}_k - (\partial_t L) dt \\ &= \textstyle\sum_k \dot{q}_k dp_k - \sum_k \dot{p}_k dq_k - (\partial_t L) dt. \end{aligned}$$

We have used that the second and forth term cancel, as $\dot{\partial}_k L = p_k$, and that, by the E-L equations, $\partial_k L = d_t \dot{\partial}_k L = d_t p_k = \dot{p}_k$.

On equating the coefficients of dp_k, and then of dq_k, we get $\dot{q}_k = \partial'_k H = \partial_{p_k} H$ and $\dot{p}_k = -\partial_k H = -\partial_{q_k} H$, respectively. And $\partial_t H = -\partial_t L$ is the the equality of the coefficients of dt .

Finally, $d_t H = \sum_k (\partial_k H)\dot{q}_k + \sum_k (\partial'_k H)\dot{p}_k + \partial_t H$, which is equal to $\partial_t H$ because the other two terms cancel ($\partial_k H = -\dot{p}_k$ and $\partial'_k H = \dot{q}_k$). □

Corollary. If L does not depend on t, then H is a conserved quantity.

Remark. Hamilton's equations form a system of $2n$ first-order ordinary differential equations in the variables $q_1, \ldots, q_n$ and $p_1, \ldots, p_n$, while the Lagrange equations form a system of n second-order ordinary differential equations in the $q_1, \ldots, q_n$. Thus Hamilton's equations can be thought of as an example of transforming a system of n second order ordinary differential equations into an equivalent system of $2n$ first order equations, with $p_1, \ldots, p_n$ in the role of "auxiliary variables".

Symmetries of the physical systems

Definitions

Examples

Noether's theorem

Examples

Let $\mathcal{X}$ be the *configuration space* (the space of the $\boldsymbol{q}$'s) of a lagrangian system Σ with lagrangian L.

A *symmetry* of Σ is a *diffeomorphism* $\varphi : \mathcal{X} \to \mathcal{X}$ such that

$$L(\varphi \boldsymbol{q}, \boldsymbol{\partial}\varphi \cdot \dot{\boldsymbol{q}}, t) = L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t),$$

where $\boldsymbol{\partial}\varphi$ is the (jacobian) gradient of φ.

A (uniparametric) *family of symmetries* is a set $\{\varphi_s\}$ of symmetries ($s \in (-\alpha, \alpha)$, $\alpha \in \mathbf{R}_{++}$) such that the map

$$(-\alpha, \alpha) \times \mathcal{X} \to \mathcal{X},\ (s, \boldsymbol{q}) \mapsto \varphi_s(\boldsymbol{q})$$

is differentiable and $\varphi_0 = \mathrm{Id}$.

If in addtion we have

$$\varphi_{s'} \circ \varphi_s = \varphi_{s+s'} \text{ when } s, s', s+s' \in (-\alpha, \alpha)\ ,$$

then we say that family is a *uniparametric group of symmetries*.

Example. If the system Σ is composed of free particles (no constraints) subject to interaction forces given by a potential V that *only depends on the distances between the particles* (Newton's gravitational potential satisfies this), then *any rigid motion is a symmetry of the system*.

Let φ be a rigid motion, say $\varphi(\boldsymbol{r}) = \tilde{\varphi}(\boldsymbol{r}) + \boldsymbol{\tau}$, where $\tilde{\varphi}$ is a linear rotation and $\boldsymbol{\tau}$ a translation vector. Then we have

$$V(\varphi\boldsymbol{r}_1, \ldots, \varphi\boldsymbol{r}_N, t) = V(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N, t),$$

because φ preserves distances and V only depends on distances.

On the other hand, $\boldsymbol{\partial}\varphi = \tilde{\varphi}$ and

$$T(\tilde{\varphi}\dot{r}_1, \ldots, \tilde{\varphi}\dot{r}_N, t) = T(\dot{r}_1, \ldots, \dot{r}_N, t),$$

because $\tilde{\varphi}$ is a linear isometry and hence $(\tilde{\varphi}\dot{r})^2 = \dot{r}^2$.

With the same notations as in the preceding example, fix $\boldsymbol{a} \in E_3$ and consider the family of translations $\varphi_s(\boldsymbol{r}) = \boldsymbol{r} + s\boldsymbol{a}$ ($s \in \mathbf{R}$). This family is a uniparametric group of symmetries of Σ.

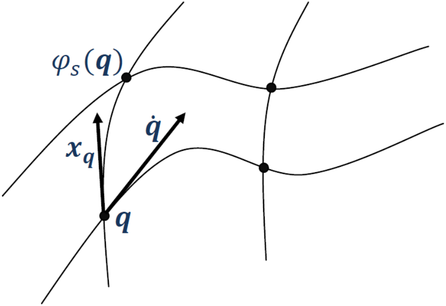
Similarly, if we let $\varphi_s(\boldsymbol{r}) = \rho_{s\boldsymbol{a}}(\boldsymbol{r})$, where $\rho_{s\boldsymbol{a}}$ is the rotation about the axis $\langle \boldsymbol{a} \rangle$ of amplitude $sa = s|\boldsymbol{a}|$, then $\{\varphi_s\}$ is an uniparametric group os symetries of Σ.

If φ_s is a family of symmetries, its *associated vector field* $\boldsymbol{x}$ is defined by the formula

$$\boldsymbol{x}_{\boldsymbol{q}} = d_s|_{s=0}(\varphi_s(\boldsymbol{q})).$$

In other words, $\boldsymbol{x}_{\boldsymbol{q}}$ is the tangent vector to the curve

$$s \mapsto \varphi_s(\boldsymbol{q}) \text{ at } \boldsymbol{q}.$$



Examples. Let $\boldsymbol{q} = (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) \in E_3^N$ and let $\varphi_s = t_{s\boldsymbol{a}}$ be the uniparametric group of translations defined before. Then it is clear that

$$\boldsymbol{x}_{\boldsymbol{q}} = (\boldsymbol{a}, \ldots, \boldsymbol{a}).$$

For the uniparametric group of rotations $\varphi_s = \rho_{s\boldsymbol{a}}$, we have

$$\boldsymbol{x}_{\boldsymbol{q}} = (\boldsymbol{a} \times \boldsymbol{r}_1, \ldots, \boldsymbol{a} \times \boldsymbol{r}_N).$$

This requires a justification:

We may choose the coordinate system so that $\boldsymbol{a} = (0, 0, a)$, $a = |\boldsymbol{a}| > 0$. Then the matrix of $\rho_{s\boldsymbol{a}}$ is

$$\begin{pmatrix} \cos(sa) & -\sin(sa) & 0 \\ \sin(sa) & \cos(sa) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The result os applying it to $\boldsymbol{r} = (x, y, z)$, followed by the derivative with respect to s at $s = 0$, yields the vector $(-ay, ax, 0)$, which is equal to $\boldsymbol{a} \times \boldsymbol{r}$. From this the claim follows immediately.

Theorem. Let φ_s be a family of symmetries of Σ and $\boldsymbol{x}$ its associated vector field. Let $\boldsymbol{p} = \dot{\partial} L$ be the canonical momenta. Then $I = \boldsymbol{p} \cdot \boldsymbol{x}$ is a conserved quantity.

Proof. By definition of symmetry, $L(\varphi_s \boldsymbol{q}, \varphi_s \dot{\boldsymbol{q}}, t) = L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$. Hence

$$
\begin{aligned}
0 &= d_{s=0} L(\varphi_s \boldsymbol{q}, \varphi_s \dot{\boldsymbol{q}}, t) \\
&= d_{s=0} L(\varphi_s \boldsymbol{q}, d_t \varphi_s \boldsymbol{q}, t) \\
&= d_{s=0} L(\boldsymbol{q} + s\boldsymbol{x} + \cdots, \dot{\boldsymbol{q}} + s\dot{\boldsymbol{x}} + \cdots, t) \\
&= d_{s=0} \big(L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) + s(\partial_{\boldsymbol{q}} L \cdot \boldsymbol{x} + \partial_{\dot{\boldsymbol{q}}} L \cdot \dot{\boldsymbol{x}}) + \cdots \big) \\
&= \partial_{\boldsymbol{q}} L \cdot \boldsymbol{x} + \partial_{\dot{\boldsymbol{q}}} L \cdot \dot{\boldsymbol{x}} \\
&= d_t(\partial_{\dot{\boldsymbol{q}}} L) \cdot \boldsymbol{x} + \partial_{\dot{\boldsymbol{q}}} L \cdot \dot{\boldsymbol{x}} \quad \text{(by E-L)} \\
&= d_t(\boldsymbol{p} \cdot \boldsymbol{x}).
\end{aligned}
$$

□

(1) We have seen that the momentum p_j of a cyclic q_j is a conserved quantity. Now we can prove this again as follows. In the $\boldsymbol{q}$-space, let $\boldsymbol{\epsilon}_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, with 1 in the j-th place. Then $\varphi_s(\boldsymbol{q}) = \boldsymbol{q} + s\boldsymbol{\epsilon}_j$ is clearly a uniparametric group of symmetries, as L does not depend on q_j. The associated vector field is $\boldsymbol{\epsilon}_j$ and Noether's conserved quantity is $\boldsymbol{p} \cdot \boldsymbol{\epsilon}_j = p_j$.

(2) *Conservation of linear momentum*. Let $\boldsymbol{a}$ be a unit vector and assume that the translations $\varphi_s = t_{s\boldsymbol{a}}$ are symmetries of Σ. We know that the conjugate momentum of $\boldsymbol{q}_j = \boldsymbol{r}_j$ is $\boldsymbol{p}_j = m_j \boldsymbol{v}_j$ and that the vector field associated to φ_s is $\boldsymbol{x} = (\boldsymbol{a}, \ldots, \boldsymbol{a})$. Noether's conserved quantity is $\sum_j \boldsymbol{p}_j \cdot \boldsymbol{a} = (\sum_j \boldsymbol{p}_j) \cdot \boldsymbol{a}$, which is the projection of the total momentum $\boldsymbol{P} = \sum_j \boldsymbol{p}_j$ on $\boldsymbol{a}$. This implies that if all translations are symmetries of Σ, then $\boldsymbol{P}$ itself is a conserved quantity.

Remark. The *center of mass*, $\boldsymbol{R}$, of the m_j is defined by $m\boldsymbol{R} = \sum_j m_j \boldsymbol{r}_j$, where $m = \sum m_j$ (*total mass*). Its velocity $\boldsymbol{V}$ satisfies $m\boldsymbol{V} = \sum_j m_j \boldsymbol{v}_j = \boldsymbol{P}$. So $\boldsymbol{V}$ is constant whenever $\boldsymbol{P}$ is a conserved quantity.

In any case, the acceleration $\ddot{\boldsymbol{R}}$ of the center of mass satisfies $m\ddot{\boldsymbol{R}} = \sum_j \boldsymbol{F}_j$.

(3) *Conservation of angular momentum*. Let $\boldsymbol{a}$ be a unit vector and assume that the rotations $\rho_{s\boldsymbol{a}}$ are symmetries of the system Σ. We know that the vector field $\boldsymbol{x}$ associated to this uniparametric group is given by, at $\boldsymbol{q} = (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N)$, by $\boldsymbol{x}_{\boldsymbol{q}} = (\boldsymbol{a} \times \boldsymbol{r}_1, \ldots, \boldsymbol{a} \times \boldsymbol{r}_N)$. The corresponding Noether conserved quantity is $\sum_j \boldsymbol{p}_j \cdot (\boldsymbol{a} \times \boldsymbol{r}_j) = \boldsymbol{a} \cdot (\sum_j \boldsymbol{r}_j \times \boldsymbol{p}_j)$, which is the projection of the *angular momentum* $\boldsymbol{L} = \sum_j \boldsymbol{r}_j \times \boldsymbol{p}_j$ on the direction $\langle \boldsymbol{a} \rangle$.

This implies that if all the rotations are symmetries, then $\boldsymbol{L}$ itself is a conserved quantity.

The Lagrange and Hamilton equations, as well as Noether's results, can be phrased intrinsically in the realm of differential geometry. Our presentation with q's and $\dot{q}$'s is the *local* treatment of the theory.

The following may be suitable texts to pursue coordinate-free approaches and delving into a myriad of related concepts and structures:

[9] (arnold-1989)
[10] (agricola-friedrich-2002)
[11] (rudolph-schmidt-2013)
[12] (rudolph-schmidt-2017).

References I

[1] M. M. Bronstein, J. Bruna, T. Cohen, and P. Velickovic, "Geometric deep learning: Grids, groups, graphs, geodesics, and gauges," 2021.
arXiv:2104.13478.

[2] T. Cohen, *Equivariant Convolutional Networks*.
PhD thesis, University of Amsterdam, 2021.

[3] C. Lavor, S. Xambó-Descamps, and I. Zaplana, *A Geometric Algebra Invitation to Space-Time Physics, Robotics and Molecular Geometry*.
SBMA/Springerbrief, Springer, 2018.

[4] T. Frankel, *The geometry of physics: an introduction*.
Cambridge University Press, 2011.

References II

[5] G. B. Folland, *Quantum Field Theory: A tourist guide for mathematicians*, vol. 149 of *Mathematical Surveys and Monographs*.

American Mathematical Soc., 2008.

[6] Y. Kosmann-Schwarzbach, *The Noether Theorems. Invariance and conservation laws in the XX century*.

Sources and Studies in the History of Mathematics and Physical Sciences, Springer, 2011.

Translation of *Les Théorèmes de Noether: Invariance et lois de conservation au XX^e^ siècle*, revised and augmented, by Bertram E. Schwarzbach.

[7] D. E. Neuenschwander, *Emmy Noether's wonderful theorem*.

The Johns Hopkins University Press, 2011.

References III

[8] J. Sivardière, *La symétrie en mathématiques, physique et chimie*.

Collection Grenoble Sciences, Presses universitaires de Grenoble, 1995.

[9] V. I. Arnol'd, *Mathematical methods of classical mechanics (2nd edition)*, vol. 60 of *Graduate Texts in Mathematics*.

Springer, 1989.

The first Russian edition appeared in 1974, the MIR French translation in 1976, and Springer's English translation in 1978. The 2nd edition has xvi+516p.

[10] I. Agricola and T. Friedrich, "Global Analysis: Differential Forms in Analysis, Geometry, and Physics," 2002.

xiv+343p.

References IV

[11] G. Rudolph and M. Schmidt, *Differential geometry and mathematical physics*.

Theoretical and Mathematicals Physics, Springer, 2013.

xiii+759p.

[12] G. Rudolph and M. Schmidt, *Differential geometry and mathematical physics: Part II. Fibre bundles, Topology and Gauge fields*.

Theoretical and Mathematicals Physics, Springer, 2017.

xvi+830p.