Embedding in law of discrete time ARMA processes in continuous time stationary processes

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Abstract

Given the stationary process X_t, $t \in \mathrm{Z}$, that satisfies the discrete ARMA model (DARMA)

$$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \sum_{k=0}^{q} \theta_k \epsilon_{t-k} \tag{1}$$

where ϵ_t is white noise with finite variance, the problem of obtaining a process satisfying a continuous version of the DARMA model (a CARMA), such that when sampled at discrete times has the same auto-covariance function as $\{X_t\}$ has been studied by several authors and termed the *embedding problem*. In [6], [8], [2] and [4] have established embeddings of some DARMA(p,q) processes in continuous ARMA(p,q), for $0 \leq q < p$. In [9] there are necessary and sufficient conditions for a DARMA process to be embedded in a CARMA process.

Brockwell in [2, 3] proposes to define CARMA processes via a state space representation of the formal equation

$$a(D)Y(t) = \sigma b(D) D\Lambda(t)$$

where $\sigma > 0$ is a scale parameter, D denotes differentiation w.r.to t, Λ is a second-order Lévy process, $a(z) = z^p + a_1 z^{p-1} + \ldots + a_p$ is a polynomial of order p and $b(z) = b_0 + b_1 + \ldots + b_q z^q$ is a polynomial of order q. The resulting CARMA is a linear function of a continuous vector autoregressive (CVAR) Markovian process.

This formalism has some limitations:

- If $q \geq p$, it requires the use of generalised processes [7].
- Even for $q < p$, not every DARMA processes are embeddable.

All these approaches to the embedding problem are only concerned with the covariance structure of the processes involved, not with their probability distributions besides the fact that, if the processes are Gaussian, the equality of the first- and second-order moments entails the equality of the probability laws. In general, the discretised version of the CARMA will not necessarily have the same law as the original DARMA. We propose a different approach to construct for any DARMA(p, q) a continuous stationary *embedding in law*. The precise statement is the following:

Theorem 1 *Given the stationary DARMA(p, q) X_t that satisfies (1) with infinitely divisible innovations ϵ_t, there exists at least one function $L : \mathbf{R}^+ \to \mathbf{R}$ decaying exponentially at infinity and a Lévy process Λ on $\mathbf{R}$, such that for each real number a the stationary processes $x_t = \int_{-\infty}^{t} L(t-s)d\Lambda(s), t \in \mathbf{R}$, sampled at times $a + t, t \in \mathbf{Z}$, have the same joint law as X_t.*

In this presentation, we shall sketch the construction of the processes x_t. Details and proofs can be found in [1].

References

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