Questions

1. Convergence/ equiconvergence of numeric series.

In various well-known tests for convergence/divergence of number series

$$\sum_{k=1}^{\infty} a_k, \tag{1}$$

with positive a_k, *monotonicity* of the sequence of $\{a_k\}$ is the basic assumption. Such series are frequently called monotone series. As examples, we mention tests by Abel, Cauchy, de la Vallee Poussin, Dedekind, Dirichlet, du Bois Reymond, Ermakov, Leibniz, Maclaurin, Olivier, Sapogov, Schlömilch; several such tests were named after Abel and Cauchy.

Typical result: the Maclaurin-Cauchy integral test. *Consider a non-negative monotone decreasing function f defined on $[1, \infty)$. Then the series*

$$\sum_{k=1}^{\infty} f(k) \tag{2}$$

converges if and only if the integral

$$\int_1^{\infty} f(t)\, dt \tag{3}$$

is finite. In particular, if the integral diverges, then the series diverges as well.

Questions : how to relax monotonicity assumption in the Maclaurin-Cauchy test and similar problems?

2. Hardy's inequality.

The following results are well known.

a). Let $a_k \geq 0, b_k \geq 0,\ \sum_{k=1}^{n} a_k = a_n \gamma_n$.

If $1 \leq p < \infty$, *then*

$$\sum_{k=1}^{\infty} a_k \Big(\sum_{n=k}^{\infty} b_n \Big)^p \leq C \sum_{k=1}^{\infty} a_k (b_k \gamma_k)^p. \tag{*}$$

If $0 < p \leq 1$, *then*

$$\sum_{k=1}^{\infty} a_k \Big(\sum_{n=k}^{\infty} b_n \Big)^p \geq C \sum_{k=1}^{\infty} a_k (b_k \gamma_k)^p. \tag{**}$$

b). Let $a_k \geq 0, b_k \geq 0,\ \sum_{k=n}^{\infty} a_k = a_n \beta_n$.

If $1 \leq p < \infty$, *then*

$$(*) \qquad \sum_{k=1}^{\infty} a_k \Big(\sum_{n=1}^{k} b_n \Big)^p \le C \sum_{k=1}^{\infty} a_k (b_k \beta_k)^p.$$

If $0 < p \le 1$, *then*

$$(**) \qquad \sum_{k=1}^{\infty} a_k \Big(\sum_{n=1}^{k} b_n \Big)^p \ge C \sum_{k=1}^{\infty} a_k (b_k \beta_k)^p.$$

Questions : is it possible to obtain analogues of (*) for $0 < p < 1$ and of (**) for $1 \le p < \infty$ under some additional conditions on a_k ?

3. Convergence of the Fourier series.

We first have to discuss integral operators. Integral transforms have their genesis in nineteenth century work of J. Fourier and O. Heaviside, subsequently set into a general framework during the twentieth century. The fundamental idea is to represent a function f in terms of a transform F, using an integral transform pair, $F(p) = \int K(p,x) f(x) dx$ and $f(x) = \int L(x,p) F(p) dp$. The functions K and L are kernels. O. Heaviside invented his operational calculus to solve differential equations, such as those arising in the theory of electrical transmission lines. The formalization of Heaviside's work leads one to the Laplace transforms $K(p,x) = e^{-px}$. One the most important integral transforms is the Fourier transform that represents functions as linear combinations of periodic functions, an idea pioneered by J. Fourier; here $K(p,x) = e^{-ipx}$.

Fourier analysis began with studying the way general functions may be represented by sums of simpler trigonometric functions. It received its name after Joseph Fourier, who showed that representing a function by a trigonometric series greatly simplifies the study of heat propagation. Today, the subject of Fourier analysis encompasses a vast spectrum of mathematics. In the sciences and engineering, the process of decomposing a function into simpler pieces is often called Fourier analysis, while the operation of rebuilding the function from these pieces is known as Fourier synthesis. The decomposition process itself is the Fourier transform.

Questions : we are interested in convergence of Fourier series and integrals with certain restriction on considered functions.

References

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