COXETER MATROIDS

JULIAN PFEIFLE

Matroids are a combinatorial structure that generalizes, for instance, the concept of families of subspaces of a vector space. One way among many to associate a matroid $\mathcal{M}$ to a configuration X of n vectors in a vector space V is to specify all subsets B of $[n] = \{1, 2, \ldots, n\}$ that index bases of V among X. Abstractly, a matroid on $[n]$ can be characterized as a system $\mathcal{M}$ of subsets of $[n]$ that satisfies *Steinitz' basis exchange axiom*:

If $A \neq B \in \mathcal{M}$ and $a \in A \setminus B$, there exists some $b \in B \setminus A$ such that $A - a + b \in \mathcal{M}$.

One way to recover geometry from this combinatorial abstraction is to work with *characteristic vectors*, by assigning to each basis B the 0/1-vector $\chi(B)$ of length n that has a '1' precisely in the coordinates indexed by B. The convex hull

$$\mathsf{MBP}(\mathcal{M}) \;=\; \operatorname{conv}\{\chi(B) : B \in \mathcal{M}\}$$

of these points is called the *matroid base polytope* of $\mathcal{M}$.

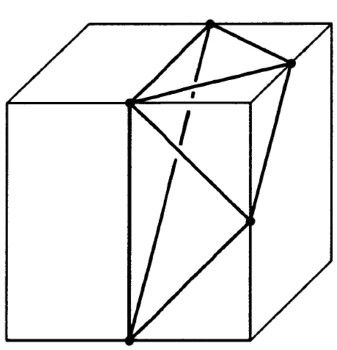
Obviously, one can study the polytope of characteristic vectors associated to any set system, but it is less than clear what, if anything, one might learn from it. However, in the case of matroids these polytopes are quite well-behaved:

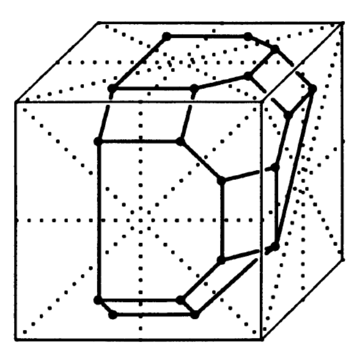
- Since all bases have the same cardinality, all characteristic vectors $\chi(B)$ lie on a sphere of radius $\sqrt{|B|}$, and therefore all of them are vertices of $\mathsf{MBP}(\mathcal{M})$.
- Edges reflect basis exchange: Two vertices $\chi(A)$, $\chi(B)$ span an edge in $\mathsf{MBP}(\mathcal{M})$ iff A, B satisfy Steinitz' axiom.

Hidden just beneath the surface of Steinitz' axiom we find the action of the *symmetric group* S_n on $[n]$: we can regard a Steinitz interchange $a \leftrightarrow b$ as the transposition (a, b), and such transpositions generate S_n. Geometrically, each edge of $\mathsf{MBP}(\mathcal{M})$ materializes the orthogonal reflection of its vertices across a hyperplane of equation $x_i = x_j$, say, and all of *those* form a very classical object: the hyperplane arrangement associated to the root system A_{n-1}.

There are various more or less abstract definitions to generalize these concepts from A_{n-1} and its associated regular polytope (the simplex) to the other classical root systems: BC_n (cubes), D_n, H_3 (dodeca/icosahedron), H_4 (120-cell and 600-cell), F_4 (24-cell), E_6, E_7, E_8, but the hands-down winner is the following charming theorem by Israel Gelfand and Vera Serganova:

Theorem-Definition (Gelfand–Serganova, 1987)
Let Q be a convex polytope. Consider, for each edge e of Q, the hyperplane H_e orthogonal to e that passes through its midpoint. Let W be the group generated by the reflections in all H_e. Then W is finite iff Q is a Coxeter matroid polytope.





Even though there's already a textbook [2] on the subject, in truth we know very little about Coxeter matroid polytopes:

- What can you say about the combinatorial types of faces of Coxeter matroid polytopes? For A_{n-1}, the only possible 2-faces are triangles and squares [2, Theorem 1.12.8], but for the other types even this very basic question seems to be open (though easy).
- What can you say about the *number* of faces of each dimension?
- Enumerate the combinatorial types of Coxeter matroid polytopes in dimension d with n vertices, for example for $d \leq 4$ and "moderate" n. A better complexity measure will probably be the size of the orbit containing the vertices of Q.
- Generalize the formula [1] for the volume of Q in the case A_{n-1} to the other cases, starting with BC_n.
- A very important construction in toric and tropical geometry is the subdivision, in the case A_{n-1}, of matroid polytopes into smaller matroid polytopes. This was first considered by Lafforgue [3]. Speyer [4] conjectured that the subdivision corresponding to *series-parallel* matroids have the largest number of faces, and he proved this in the case of tropical linear spaces. What can you say about subdivisions of Coxeter matroid polytopes into smaller Coxeter matroid polytopes?
- Generalize operations on matroids, such as duality, contraction, extension and deletion, from A_{n-1} to the other cases.

Obviously, we won't try to answer all of these questions in two months, or all the new ones we'll come across in our investigations!

However, the prerequisites being quite modest (essentially, having read chapter 6 of [2] and whatever else you need to more or less understand that material), we will be able to dedicate most of our time to research. For visualization and calculations you can use software such as `polymake`, `sage` or `gap`.

References

[1] Federico Ardila, Carolina Benedetti, and Jeffrey Doker. Matroid polytopes and their volumes. *Discrete Comput. Geom.*, 43(4):841–854, 2010.
[2] Alexandre V. Borovik, I. M. Gelfand, and Neil White. *Coxeter matroids. With illustrations by Anna Borovik.* Boston, MA: Birkhäuser, 2003.
[3] L. Lafforgue. *Chirurgie des grassmanniennes.* Providence, RI: American Mathematical Society (AMS), 2003.
[4] David E. Speyer. Tropical linear spaces. *SIAM J. Discrete Math.*, 22(4):1527–1558, 2008.

Dept. Matemàtiques, Universitat Politècnica de Catalunya
E-mail address: `julian.pfeifle@upc.edu`