Cobordism Theory: Old and New

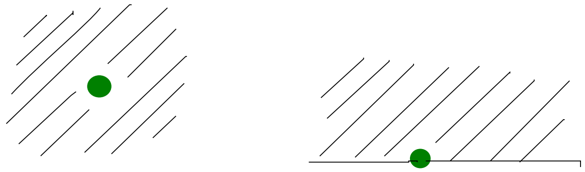
Ulrike Tillmann, Oxford

2016 BGSMath Scientific Meeting – Barcelona

Manifolds

M is a *manifold of dimension d* if locally it is diffeomorphic to

$$\mathbb{R}^d \quad \text{or} \quad \mathbb{R}^d_{\geq 0}.$$



M is *closed* if it is compact and has no boundary.

Fundamental problem:

– classify compact smooth manifolds M of dim d;

– understand their group of diffeomorphisms Diff(M).

d any : the empty $\varnothing$ set is a manifold of any dimension

$d = 0$: M is a collection of finitely many points

$d = 1$: M is a collection of circles S^1 and intervals $[0, 1]$

$d = 2$: M is a collection of orientable surfaces $F_{g,n}$ and non-orientable surfaces $N_{g,n}$ of genus g and with n boundary components

Leitmotif = Classification of Manifolds

1. Classical Cobordism Theory (Thom, ...)

2. Topological Field Theory (Witten, Atiyah, Segal, ...)

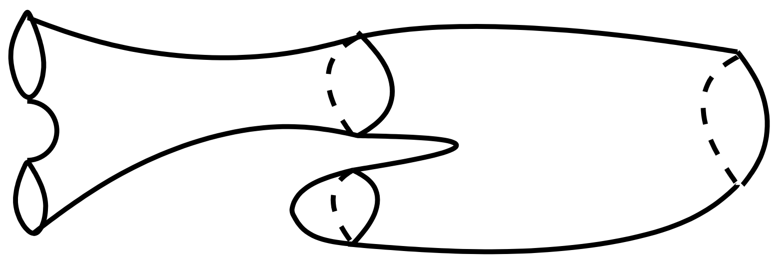
3. Cobordism Hypothesis (Baez-Dolan, Lurie, ...)

4. Classifying spaces of cobordism categories

—— Mumford conjecture (Madsen-Weiss, ...)
—— Classification of invertible theories (GMTW)
—— Filtration of the classical theory

1. Classical Cobordism Theory

Definition: Two closed oriented $(d-1)$-dimensional manifolds M_0 and M_1 are *cobordant* if there exists a compact oriented d-dimensional manifold W with boundary $\partial W = \bar{M}_0 \sqcup M_1$.

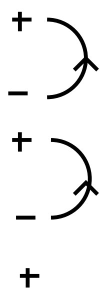


$$M_0 \longrightarrow W \longleftarrow M_1 \longrightarrow W' \longleftarrow M_2$$

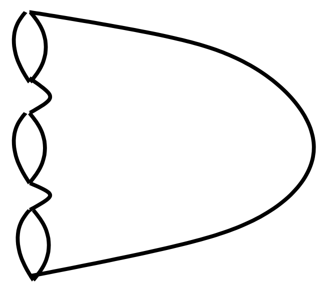
- *equivalence relation*; equivalence classes $=: \mathfrak{N}^+_{d-1}$
- group with *product* $\coprod$ and *inverse* $M^{-1} = \bar{M}$;
- graded ring $\bigoplus_{d>0} \mathfrak{N}^+_{d-1}$ with *multiplication* $\times$.

Examples:

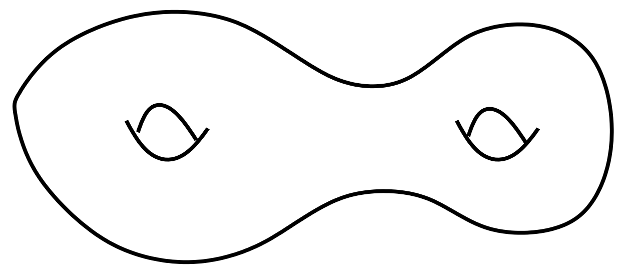
$\mathfrak{N}_0^+ = \mathbb{Z}$



$\mathfrak{N}_1^+ = \{0\}$



$\mathfrak{N}_2^+ = \{0\}$



Theorem (Thom) $\mathfrak{N}_d^+ = \pi_d(\Omega^\infty \mathbf{MSO})$

where $\Omega^\infty \mathbf{MSO} := \lim_{n\to\infty} \lim_{k\to\infty} \text{maps}_*(S^n, (U_{n,k})^c)$

and $U_{n,k} \to Gr^+(n,k)$ is the universal n-dimensional bundle over the Grassmannian manifold of oriented n-planes in $\mathbb{R}^{n+k}$.

$M \subset$ tubular neighbourhood $N(M) \subset \mathbb{R}^{d+n}$

$$\mathsf{S}^{d+n} = (R^{d+n})^c \xrightarrow{collapse} (N(M))^c \xrightarrow{\phi_{N(M)}} (U_{n,d})^c$$

$$. \qquad (x,v) \mapsto (N_x M, v).$$

Theorem (Thom) $\mathfrak{N}_*^+ \otimes \mathbb{Q} \simeq \mathbb{Q}\,[\mathbb{C}P^2, \mathbb{C}P^4, \ldots]$.

Proof: For fixed $*$ and large n and k,

$$\begin{aligned}
\pi_*(\Omega^\infty \mathbf{MSO}) \otimes \mathbb{Q} &= \pi_*(\lim_{n\to\infty} \lim_{k\to\infty} \text{maps}_*(S^n, (U_{n,k})^c)) \otimes \mathbb{Q} \\
&= \pi_*(\text{maps}_*(S^n, (U_{n,k})^c)) \otimes \mathbb{Q} \\
&= \pi_{*+n}((U_{n,k})^c) \otimes \mathbb{Q} \\
&= H_{*+n}((U_{n,k})^c) \otimes \mathbb{Q} \quad \text{by Serre} \\
&= H_*(Gr^+(n,k)) \otimes \mathbb{Q} \quad \text{by Thom.}
\end{aligned}$$

Réné Thom (1923–2002); Fields Medal 1958

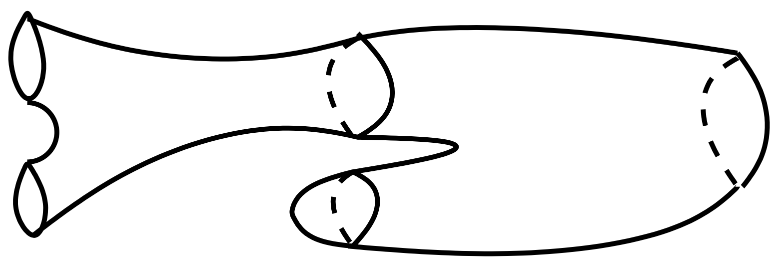
2. Topological Field Theory

$\mathcal{C}ob_d^\delta$ is the *discrete cobordism category* with

Objects: closed oriented $d-1$ dimensional manifolds M

Morphisms from M_0 to M_1: equivalence classes of d-dimensional cobordism W with $\partial W = \bar{M}_0 \sqcup M_1$ w.r.t. diffeomorphisms rel. boundary

Composition: gluing of cobordisms.



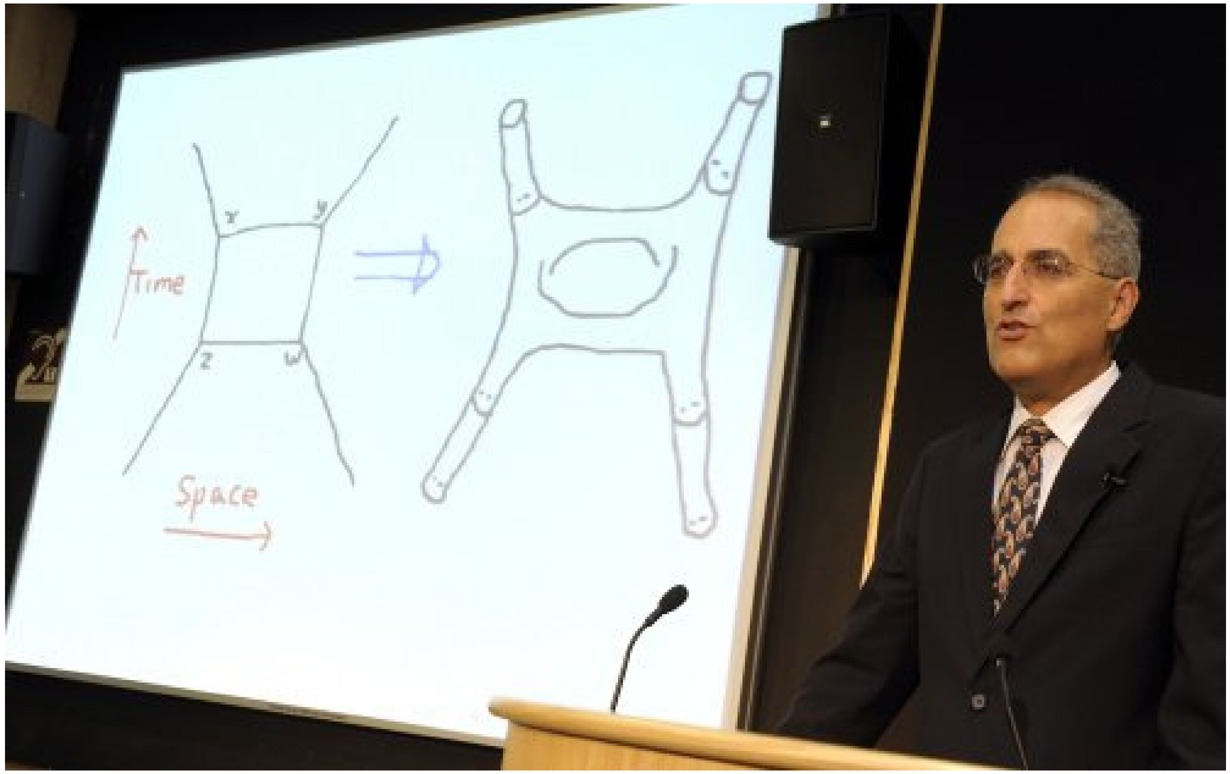
$$W' \circ W : M_0 \longrightarrow M_1 \longrightarrow M_2$$

Definition: A *d-dimensional TFT* is a functor

$$\mathcal{F} : \mathcal{C}ob_d^{\delta} \longrightarrow \mathcal{V}$$

to the category $\mathcal{V}$ of vector spaces that takes disjoint union of manifolds to tensor products of vector spaces.

Example: $\mathcal{F}(\varnothing) = \mathbb{C}$.



Edward Witten

Michael Atiyah

Motivation:

d-dimensional TFTs define topological invariants for d-dimensional closed manifolds:

If $\partial W = \varnothing$ then it defines a morphisms $W : \varnothing \to \varnothing$, and $\mathcal{F}$ assigns a number to W depending only on its topology:

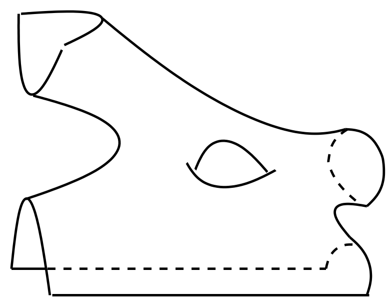
$$\mathcal{F}(W) : \mathcal{F}(\varnothing) = \mathbb{C} \longrightarrow \mathcal{F}(\varnothing) = \mathbb{C}$$

Physical inspiration: locality!

3. Cobordism Hypothesis

Categorification:
points, cobordisms, cobordisms of cobordisms, ...

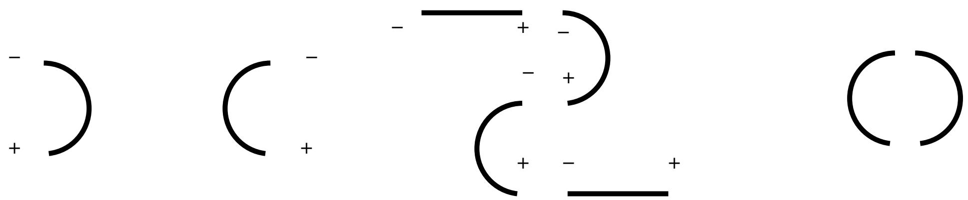
$\mathcal{C}ob_d^\delta$ replaced by d-fold category $ex\mathcal{C}ob_d^\delta$
$\mathcal{V}$ replaced by a d-fold symmetric monoidal category $\mathcal{V}_d$
extended TFTs: $ex\mathcal{C}ob_d^\delta \xrightarrow{\mathcal{F}} \mathcal{V}_d$.



Cobordism hypothesis (Baez-Dolan)
Extended TFTs are determined by $\mathcal{F}(*)$.

Example: 1-dimensional theories:

Let $\mathcal{F}(*_+) = V$ and $\mathcal{F}(*_-) = V'$.



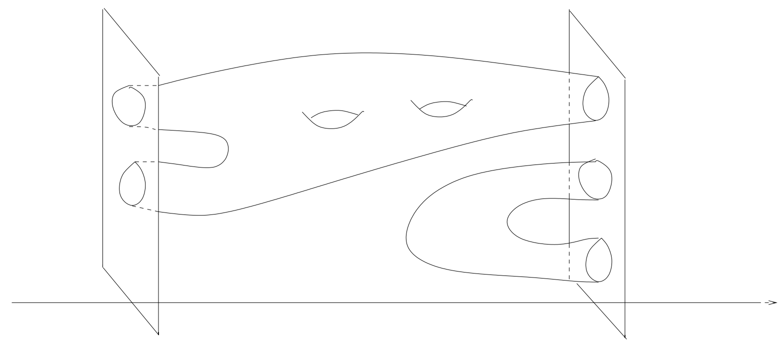
- evaluation $e : V \otimes V' \to \mathbb{C}$
- co-evaluation $e^* : \mathbb{C} \to V' \otimes V$
- V is finite dimensional as

$$id : V \xrightarrow{id \otimes e^*} V \otimes V' \otimes V \xrightarrow{e \otimes id} V$$

- $e \circ e^* = \text{dim}(V) : \mathbb{C} \to \mathbb{C}$

Enriched TFTs

Consider moduli spaces of all compact $(d-1)$- and d-manifolds embedded in $\mathbb{R}^{d+n}, n \to \infty$, to form the topological category $\mathcal{C}ob_d$.



$$M_0 \longrightarrow W \longleftarrow M_1$$

The homotopy type of the space of morphisms is

$$mor_{\mathcal{C}ob_d}(M_0, M_1) \simeq \coprod_W B\text{Diff}(W;\partial)$$

where the disjoint union is taken over all diffeomorphism classes of cobordisms W.

Theorem (Hopkins-Lurie ($n = 2$), **Lurie** (general)**)**:

$$\mathcal{F} : ex\mathcal{C}ob_d^{fr} \to \mathcal{V}_d$$

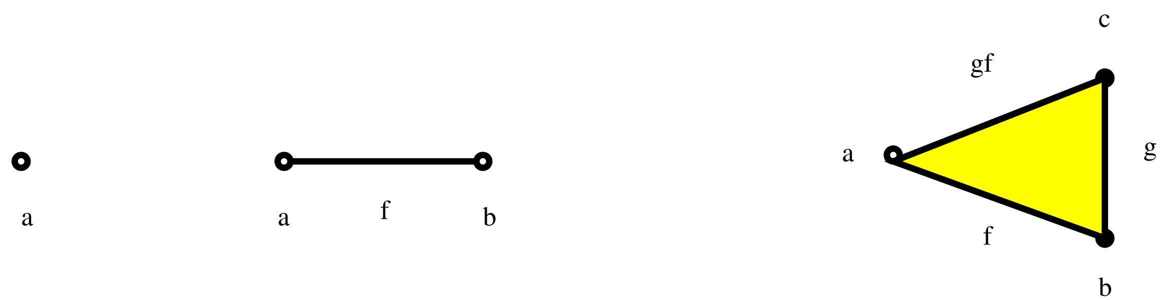
is determined by $\mathcal{F}(*)$, the value on a point.
Vice versa, any object in $\mathcal{V}_d$ satisfying certain duality and non-degeneracy properties gives rise to a TFT.

Here: W is *framed* if its stable normal bundle is trivial.

More general: for non-orientable, oriented, ..., $\mathcal{F}$ is still determined by $\mathcal{F}(*)$ but there are group actions that have to be considered.

4. Classifying space of cobordism categories

B : Topological Categories $\longrightarrow$ Spaces, $\qquad \mathcal{C} \mapsto B\mathcal{C}$



- morphisms $\mapsto$ paths *which are homotopy invertible!*

. for every $a \in ob_{\mathcal{C}}$, there is a characteristic map

$$\alpha : mor_{\mathcal{C}}(a,a) \longrightarrow \text{maps}([0,1],\partial; B\mathcal{C}, a) = \Omega B\mathcal{C}$$

- monoidal cats $\mapsto$ E_1-spaces (Ω-spaces)
- symmetric monoidal cats $\mapsto$ E_∞-spaces (Ω^∞-spaces)

Theorem (Galatius, Madsen, Tillmann, Weiss)

$$\Omega B(\mathcal{C}ob_d) \simeq \Omega^\infty \mathbf{MTSO}(d) = \lim_{n\to\infty} \Omega^{d+n}((U_{d,n}^\perp)^c)$$

where $U_{d,n}^\perp$ is the orthogonal complement of the universal bundle $U_{d,n} \to Gr^+(d,n)$.

Note: the Thom class is in dimension $-d$!

The characteristic map:
mor$_{\mathcal{C}ob_d}(\varnothing,\varnothing) \ni W \subset N(W) \subset \mathbb{R}^{d+n}$,

$$\alpha(W) : S^{d+n} = (R^{d+n})^c \xrightarrow{collapse} N(W)^c \xrightarrow{\phi_{T(W)}} (U_{d,n}^\perp)^c$$

$$(x,v) \mapsto (T_x W, v).$$

In Thom's theory: $(x,v) \mapsto (N_x W, v) \in (U_{n,d})^c$.

$$H^*(\Omega_0^\infty \mathbf{MTSO}(d), \mathbb{Q}) \simeq \mathbf{\Lambda}^*(H^{*>0}(BSO(d); \mathbb{Q})[-d])$$

Theorem (Barrett-Priddy, Quillen, Segal)
For $d = 0$: $B\Sigma_n \xrightarrow{\alpha} \Omega^\infty \mathbf{MTSO}(0) \simeq \Omega^\infty S^\infty$ is a homology isomorphism in degrees $* \leq n/2$.

Theorem (Madsen-Weiss)
For $d = 2$: $B\mathrm{Diff}(F_g) \xrightarrow{\alpha} \Omega^\infty \mathbf{MTSO}(2)$ is a homology isomorphism in degrees $* \leq (2g-2)/3$.

$\Longrightarrow$ **Mumford's Conjecture**:

$$H^*(\mathcal{M}_g; \mathbb{Q})) \simeq H^*(B\mathrm{Diff}(F_g), \mathbb{Q}) \sim \mathbb{Q}[\kappa_1, \kappa_2, \ldots]$$

Galatius & Randal-Williams prove analogue in higher dimensions for connected sums of $S^d \times S^d$, $d > 2$.

Filtration of classical cobordism theory

The inclusion of multi-categories

$$ex\mathcal{C}ob_1 \subset \cdots \subset ex\mathcal{C}ob_{d-1} \subset ex\mathcal{C}ob_d \subset \ldots$$

induces on taking multi-classifying spaces a filtration

$$\Omega^\infty S^\infty \to \cdots \to \Omega^{\infty-(d-1)}\mathbf{MTSO}(d{-}1) \to \Omega^{\infty-d}\mathbf{MTSO}(d)\ldots$$

of Thom's space $\Omega^\infty\mathbf{MSO}$ which respects the additive and multiplicative structure.
All Thom classes are in degree zero!

For framed manifolds, this is the constant filtration

$$B(ex\mathcal{C}ob_d^{fr}) = \lim_{n\to\infty} \Omega^n(\tilde{U}_{d,n}^{\perp})^c \simeq \Omega^\infty S^\infty$$

where $\tilde{U}_{d,n}$ is the universal bundle over the Stiefel manifold of framed d-planes in $\mathbb{R}^{d+n}$.

Cobordism Hypothesis for invertible theories

An extended framed TFT

$$\mathcal{F} : ex\mathcal{C}ob_d^{fr} \longrightarrow \mathcal{V}_d$$

induces a map of infinite loop spaces

$$B\mathcal{F} : B(ex\mathcal{C}ob_d^{fr}) \simeq \Omega^\infty S^\infty \longrightarrow B(\mathcal{V}_d).$$

$\Omega^\infty S^\infty$ is the free infinite loop space on one point.
$\Longrightarrow$ *$B\mathcal{F}$ is determined by its value on that point, $B\mathcal{F}(*)$.*

If $\mathcal{F}$ is *invertible* (in the sense that the images of all morphisms are invertible) it *factors through $B\mathcal{F}$*.

Fibration sequence

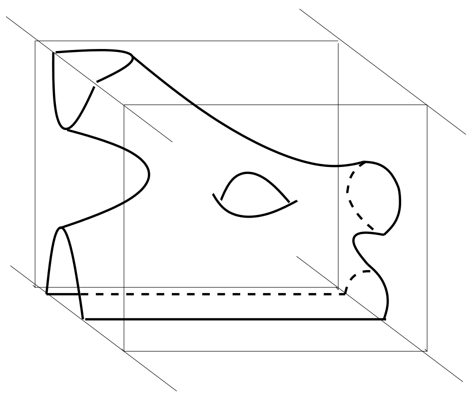
$$\Omega^\infty \mathbf{MTSO}(d) \longrightarrow \Omega^\infty \Sigma^\infty (BSO(d)_+) \longrightarrow \Omega^\infty \mathbf{MTSO}(d-1).$$

Genauer proves that this corresponds to natural maps of cobordism categories:

$\mathcal{C}ob_d$: d-dim cobordisms in $[a_0, a_1] \times \mathbb{R}^{d+n-1} \times (0, \infty)$
$\bigcap$
$\mathcal{C}ob_d^\partial$: d-dim cobordisms in $[a_0, a_1] \times \mathbb{R}^{d+n-1} \times [0, \infty)$
$\downarrow$
$\mathcal{C}ob_{d-1}$: $d-1$-dim cobordisms in $[a_0, a_1] \times \mathbb{R}^{d-1+n} \times \{0\}$

An even finer filtration

$$\Omega^\infty \mathbf{MSO} \simeq \lim_{n\to\infty} \lim_{d\to\infty} \Omega^n (U_{n,d})^c$$
$$\simeq \lim_{d\to\infty} \lim_{n\to\infty} \Omega^n (U^\perp_{d,n})^c$$
$$\simeq \lim_{d\to\infty} \lim_{n\to\infty} B(\mathcal{C}ob^d_{d,n})$$



A 2-morphism in $\mathcal{C}ob^2_{2,1}$.