PROPOSAL OF BGSM'S COURSE: *SYMPLECTIC TECHNIQUES IN DYNAMICAL SYSTEMS AND MATHEMATICAL PHYSICS*

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Abstract. The aim of this course is to study the geometry of Symplectic manifolds and to provide geometrical tools to tackle problems in Hamiltonian Dynamics.

1. Motivation

Symplectic[1] geometry studies manifolds endowed with a closed non-degenerate 2-form. The non-degeneracy of this 2-form allows to associate vector fields to functions (Hamiltonian vector fields). This assignment is key in this theory. The study of symplectic manifolds shows up naturally in Hamiltonian systems. The very Hamiltonian equations can be seen as the equations of the flow of Hamiltonian vector field associated to the energy (Hamiltonian).

In contraposition to Riemannian manifolds, Symplectic manifolds have no local invariants (Darboux theorem implies that locally all Symplectic manifolds look alike) but the study of global invariants or of local invariants with additional structures (group actions, integrable systems, Lagrangian manifolds) make this theory into a fascinating subject that places itself in the crossing roads of Geometry, Topology and Dynamics.

We propose a 60-hours course in Symplectic Geometry and Topology with an eye towards applications in Hamiltonian Dynamics. The first part of the course in classical Symplectic Geometry settles the language and tools of the theory, it also contains a chapter on integrable Hamiltonian systems, a chapter on the study of Symmetries which special attention to the abelian case (study of toric manifolds also of interest for algebraic geometers) and a final chapter on the study of Poisson Geometry which is a natural generalization of Symplectic manifolds. In the final part of the course, the theory of intersection of Lagrangian submanifolds is applied to the invariant manifolds of invariant objects

[1]The word *symplectic* was coined by Weyl as a literal Greek translation of the Latin word *complexus*; prior to the invention of this word, what is nowadays called symplectic geometry received different names, some of them involving the word complex. Weyl pretended to avoid confusion with complex geometry while preserving a reference to the relation of symplectic geometry to complex geometry.

of Hamiltonian systems, like equilibrium points, periodic orbits, invariant tori, the Poisson systems are introduced in a natural way in Celestial Mechanics close to the collision or infinite manifolds, and the principle of minimal action is broadly applied in several settings, like billiards, elliptic fixed points of area-preserving maps, and positive definite invariant tori.

The course has a multidisciplinary approach and will therefore be of interest to both Geometers (Differential and algebraic) and Dynamicists, as well as appealing to graduate students working with partial differential equations. In particular, the interaction between graduate students from these different areas will be promoted. There is also a high intersection of interests with Physicists, indeed, depending on the audience, other topics such as Geometric Quantization could be covered as well.

1.1. **Motivating example.**

Take a holomorphic function on $\mathbb{C}^2$, $F : \mathbb{C}^2 \longrightarrow \mathbb{C}$ decompose it as $F = G + iH$ with $G, H : \mathbb{R}^4 \longrightarrow \mathbb{R}$.

Cauchy-Riemann equations for F in coordinates $z_j = x_j + iy_j$, $j = 1, 2$ read as,

$$\frac{\partial G}{\partial x_i} = \frac{\partial H}{\partial y_i}, \quad \frac{\partial G}{\partial y_i} = -\frac{\partial H}{\partial x_i}$$

- We can reinterpret these equations as the equality

$$\{G, \cdot\}_0 = \{H, \cdot\}_1 \quad \{H, \cdot\}_0 = -\{G, \cdot\}_1$$

  with $\{\cdot, \cdot\}_j$ the Poisson brackets associated to the real and imaginary part of the symplectic form $\omega = dz_1 \wedge dz_2$ ($\omega = \omega_0 + i\omega_1$).
- Both Poisson structures defined by the real and imaginary part of the symplectic form are compatible.
- It is easy to check that $\{G, H\}_0 = 0$ and $\{H, G\}_1 = 0$. In other words, the real and imaginary components of the holomorphic function define an integrable system with respect to both Poisson (indeed symplectic!) structures considered above.
- This example shows the connection between known equations like Cauchy Riemann and the contents of this course.

2. The contents

This course consists of three parts which have as *driving force* the list of problems proposed by Arnold in his seminal paper [2] and, more recently, by Bramham and Hofer in [3].

The first part called **Symplectic Geometry and Hamiltonian actions**, a second part called **Symplectic Topology** and a last part called **Applications.** The lecturers of these three parts will be Eva Miranda, Ignasi Mundet and Amadeu Delshams. Eva Miranda will act as a coordinator of the course.

(1) **Part I: Symplectic Geometry and Hamiltonian actions,** *Lecturer: Eva Miranda.*
   (a) (4 hours) Basics in Symplectic Geometry. Lagrangian submanifolds. Moser's path method. Darboux theorem. Normal forms.
   (b) (2 hours) Group actions on a manifold. Symplectic and Hamiltonian group actions on a Symplectic Manifold.
   (c) (4 hours) Integrable Hamiltonian systems: Arnold-Liouville theorem. Some generalizations.
   (d) (4 hours) Toric manifolds. Delzant theorem.
   (e) (4 hours) Introduction to Poisson Geometry. Symplectic foliation. Weinstein's splitting theorem. Normal forms in Poisson Geometry.
   (f) (2 hours) Integrable Hamiltonian Systems in Poisson manifolds.
   (g) (Tentative)(4 hours) A crash course on Geometric Quantization.

(2) **Part II: Symplectic Topology,** *Lecturer: Ignasi Mundet.*
   (a) (2 hours) Almost complex structures and pseudoholomorphic curves.
   (b) (1 hour) Review of Sobolev spaces and Fredholm operators.
   (c) (3 hours) Elliptic operators, Garding's inequality and Riemann–Roch.
   (d) (2 hours) Local and cylindrical estimates for pseudoholomorphic curves.
   (e) (3 hours) Gromov compactness.
   (f) (1 hour) Symplectic nonsqueezing.
   (g) (1 hour) Review of Morse theory and the Morse–Smale–Witten complex.
   (h) (3 hours) Floer homology.
   (i) (2 hours) Arnold's conjecture.
   (j) (2 hours) Hofer's norm on the group of Hamiltonian symplectomorphisms.

(3) **Part III: Applications to Dynamical systems,** *Lecturer: Amadeu Delshams.*
   (a) (3 hours) Intersection theory of Lagrangian invariant manifolds of invariant objects in Hamiltonian systems.
   (b) (3 hours) Perturbative setting: Splitting potential and Poincaré-Arnold function, Melnikov potential.
   (c) (3 hours) Symplectic normally hyperbolic invariant manifolds in Hamiltonian Systems, Scattering maps.
   (d) (3 hours) Poisson structures. Applications to the Infinity and Collision Manifolds in Celestial Mechanics.

(e) (3 hours) Minimal Action in convex billiards (length spectrum: Can one hear the shape of a drum?), periodic orbits, caustics, integrability and non-integrability.
(f) (3 hours) The minimal action in an area-preserving map near a general elliptic fixed point, Birkhoff invariants. Applications to closed geodesics of surfaces.
(g) (2 hours) The minimal action near positive definite invariant tori, Birkhoff normal form.

References

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[2] V. I. Arnold, *Symplectic Geometry and Topology*, J. Math. Phys. 41, 3307 (2000).
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[4] V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics*, Cambridge University Press, 1990.
[5] Dusa McDuff and D. Salamon, *Introduction to Symplectic Topology*, Oxford University Press, 1998. ISBN 0-19-850451-9.
[6] K. R. Meyer, G. R. Hall, D. Offin, *Introduction to Hamiltonian dynamical systems and the N-body problem.* Second edition. Applied Mathematical Sciences, 90. Springer, New York, 2009.
[7] L. Polterovich, *The Geometry of the Group of Symplectic Diffeomorphisms*, Lectures in Mathematics ETH Zrich. Birkhuser Verlag, Basel, 2001.
[8] K. F. Siburg, *The Principle of Least Action in Geometry and Dynamics.* Lecture Notes in Mathematics, 1844. Springer-Verlag, Berlin, 2004.

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